18.06 MIDTERM 2

November 1, 2019 (50 minutes)

Please turn cell phones off completely and put them away.

No books, notes, or electronic devices are permitted during this exam.

You must show your work to receive credit. **JUSTIFY EVERYTHING**.

Please write your name on **ALL** pages that you want graded (those will be the ones we scan).

The back sides of the paper will **NOT** be graded (for scratch work only).

Do not unstaple the exam, nor reorder the sheets.

Problem 1 has $\boxed{5}$ parts, Problem 2 has $\boxed{3}$ parts, Problem 3 has $\boxed{3}$ parts.

NAME: ______________________________

MIT ID NUMBER: ______________________________

RECITATION INSTRUCTOR: ______________________________

PROBLEM 1 **NAME:** ____________________

(1) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be linearly independent vectors in $\mathbb{R}^n$. What is the rank of the matrix:

$$A = \begin{bmatrix}\mathbf{v}_1 | \mathbf{v}_2\end{bmatrix}$$

whose columns are the given vectors? *(5 pts)*
Solution: Note that the column space is spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, but since these are linearly independent as stated in the question, this gives a basis of the column space. So $rank(A) = dim(C(A)) = 2$ as it has a basis of 2 vectors.

(2) For any vector $\mathbf{b} \in \mathbb{R}^n$, its projection onto the subspace V spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is:

$$\mathrm{proj}_V \mathbf{b} = P_V \mathbf{b} \qquad \text{where the projection matrix is} \qquad \boxed{P_V = A(A^T A)^{-1} A^T}$$

Use this to obtain a formula (in terms of A and $\mathbf{b}$) for the real numbers α and β defined by the property that $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2$ is the closest vector in the subspace V to the vector $\mathbf{b}$. *(10 pts)*
Solution: Denote by $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = (A^T A)^{-1} A \mathbf{b}$. With this definition we have

$$\mathrm{proj}_V \mathbf{b} = A \begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2$$

by using the definition of P_V given above. Thus the above given formula for α and β as required from the problem.

(3) Explain why, for any given vector $\mathbf{b}$, the numbers α and β in (2) are unique. *(5 pts)*
Solution: Assume we have also a solution α' and β' still describing the projection vector as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, ie we have

$$\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = \alpha' \mathbf{v}_1 + \beta' \mathbf{v}_2$$

and so we can rewrite this equation as

$$(\alpha - \alpha')\mathbf{v}_1 + (\beta - \beta')\mathbf{v}_2 = 0$$

But this gives a 0 linear combination of linearly independent vectors, so by definition of linearly independence, the coefficients of this linear combination are all 0 or in other words

$$\alpha = \alpha'$$
$$\beta = \beta'$$

Thus we see the coefficients α, β are unique.

(4) Let $\mathbf{v}_1$, $\mathbf{v}_2$, A still be as on the previous page, and consider the vectors $\mathbf{w}_1 = 3\mathbf{v}_1 - 2\mathbf{v}_2$ and $\mathbf{w}_2 = 2\mathbf{v}_1 - \mathbf{v}_2$. We consider the matrix whose columns are these new vectors:

$$B = \left[\mathbf{w}_1 | \mathbf{w}_2\right]$$

Decide whether $B = AX$ or $B = XA$ for some matrix X. What is X? Explain. *(5 pts)*
Solution: We can write

$$B = AX = A \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$$

This is because AX changes A by taking linear combinations of columns, while XA changes it by taking linear combination of rows and by definition B is constructed from A by taking linear combinations of the columns.

(5) The projection matrix onto the subspace W spanned by $\mathbf{w}_1$ and $\mathbf{w}_2$ is:

$$\boxed{P_W = B(B^TB)^{-1}B^T}$$

Prove that $P_W = P_V$, either by a geometric argument, or by a computation. *(10 pts)*

Solution:

- geometric: Note that $\mathbf{w}_1$ and $\mathbf{w}_2$ span the same vector space as $\mathbf{v}_1$ and $\mathbf{v}_2$, since $\mathbf{v}_1 = -\mathbf{w}_1 + 2\mathbf{w}_2$ and $\mathbf{v}_2 = -2\mathbf{w}_1 + 3\mathbf{w}_2$. Thus $W = V$ and so the matrices P_V and P_W are the orthogonl projection matrices to the same subspace and hence they are exactly the same matrix as required.

- algebraic:

$$P_W = B(B^TB)^{-1}B^T = AX(X^TA^TAX)^{-1}X^TA =$$
$$AXX^{-1}(A^TA)^{-1}(X^T)^{-1}X^TA^T = A(A^TA)^{-1}A^T = P_V$$

PROBLEM 2 (1) Use Gram-Schmidt to obtain a factorization (show all your steps):

$$\boxed{A = QR} \qquad \text{of the matrix} \qquad A = \begin{bmatrix} 1 & 6 \\ 4 & 15 \\ 8 & 12 \end{bmatrix}$$

where Q has orthonormal columns and R is an upper triangular square matrix. *(15 pts)*
Solution: Using Gram-Schmidt we first rescale the first column to get

$$\mathbf{q}_1 = \mathbf{v}_1/\|\mathbf{v}_1\| = \frac{1}{9}\begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

Then we compute an orthogonal vector in the span of the two to get

$$\mathbf{q}_2' = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{q}_1)\mathbf{q}_1 = \mathbf{v}_2 - 18\mathbf{q}_1 = \begin{bmatrix} 4 \\ 7 \\ -4 \end{bmatrix}$$

and normalizing we get

$$\mathbf{q}_2 = \frac{1}{9}\begin{bmatrix} 4 \\ 7 \\ -4 \end{bmatrix}$$

Thus we can write the factorization as

$$A = QX = \frac{1}{9}\begin{bmatrix} 1 & 4 \\ 4 & 7 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} 9 & 18 \\ 0 & 9 \end{bmatrix}$$

(2) With the notation as on the previous page, consider the linear transformation:

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(\mathbf{v}) = Q\mathbf{v}$$

Suppose you have any two orthogonal (i.e. perpendicular) vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$. Prove that the vectors $f(\mathbf{v}_1), f(\mathbf{v}_2) \in \mathbb{R}^3$ are also orthogonal (justify all your steps). *(10 pts)*

Solution: By assumption we know $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1^T\mathbf{v}_2 = 0$. Also as the columns of Q are orthonormal we know that $Q^TQ = I_2$ the identity matrix, thus we get

$$f(\mathbf{v}_1) \cdot f(\mathbf{v}_2) = f(\mathbf{v}_1)^T f(\mathbf{v}_2) = (Q\mathbf{v}_1)^T(Q\mathbf{v}_2) = \mathbf{v}_1^T Q^T Q \mathbf{v}_2 = \mathbf{v}_1^T\mathbf{v}_2 = 0$$

Thus $f(\mathbf{v}_1)$ and $f(\mathbf{v}_2)$ are orthogonal

(3) Compute an eigenvector $\mathbf{a}$ of the matrix R and the corresponding eigenvalue *(Hint: it's easy to spot the eigenvector just by looking at the matrix R).* Draw the linear transformation:

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \qquad g(\mathbf{w}) = R\mathbf{w}$$

on a picture of $\mathbb{R}^2$, by drawing the eigenvector $\mathbf{a}$ and showing where the function g sends $\mathbf{a}$ and any other vector in $\mathbb{R}^2$ of your choice, linearly independent from $\mathbf{a}$. *(10 pts)*

Solution: Note that R is upper triangular so $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is always an eigenvector. In particular note in this case that $Re_1 = 9e_1$.

To describe the transformation it is given by stretching the vector e_1 by 9 and a vector not in the direction of e_1 is stretched by 9 and then translated in the direction of e_1

PROBLEM 3

(1) Assume a, d, f are non-zero numbers and b, c, e are arbitrary. Compute all 9 cofactors of:

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

and use them to obtain a formula for the inverse matrix A^{-1}. You may use the well-known formula for 2×2 determinants $\det \begin{bmatrix} x & y \\ z & t \end{bmatrix} = xt - yz$. *(10 pts)*

Solution: Putting the cofactors in the corresponding matrix entry, using the 2x2 determinant we compute

$$\begin{bmatrix} df & 0 & 0 \\ bf & af & 0 \\ be - cd & ac & ad \end{bmatrix}$$

Now to compute A^{-1} we need to add signs take the transpose and divide by the determinant. The only thing hence missing to compute is the determinant of A. But as A is upper triangular $det(A) = adf$ the product of the diagonal entries, hence

$$A^{-1} = \frac{1}{adf} \begin{bmatrix} df & -bf & be - cd \\ 0 & af & -ae \\ 0 & 0 & ad \end{bmatrix}$$

(2) Explain why all $5! = 120$ terms in the big formula for the determinant:

$$\begin{bmatrix} 0 & 0 & 0 & a_{14} & a_{15} \\ 0 & 0 & 0 & a_{24} & a_{25} \\ 0 & 0 & 0 & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

are zero. *(10 pts)*

Solution: Note that each term uses exactly 1 entry in each row and each column. So it has to use 3 entries in the first 3 columns. But note that if you have any entry in the first 3 rows of the first 3 columns you get 0. So we of the first 3 columns we need to use all entries in the last 2 rows. But you can't pick 3 entries of the last 2 rows, such that they are all in diferent rows.

(3) Use row operations to compute the determinant of the matrix:

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{bmatrix}$$

(if instead of row operations, you use the formula for 3×3 determinants as a sum of 6 terms to compute the above, you will lose at least half of the points). *(10 pts)*

Solution: We apply Gaussian elimination to get

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 0 & \beta - \alpha & (\beta - \alpha)(\beta + \alpha) \\ 0 & \gamma - \alpha & (\gamma - \alpha)(\gamma + \alpha) \end{bmatrix}$$

If $\beta \neq \alpha$ we can divide the second row by $\beta - \alpha$ and multiply by $\gamma - \alpha$ and take this away from the last row to get

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ 0 & \beta - \alpha & (\beta - \alpha)(\beta + \alpha) \\ 0 & 0 & (\gamma - \alpha)(\gamma - \beta) \end{bmatrix}$$

So as this is upper triangular we can compute the determinant by taking the product of the diagonal entries to get $det(A) = (\beta - \alpha)(\gamma - \alpha)(\gamma - \beta)$

If $\beta = \alpha$ we have after the second step the second row is 0, so the determinant is $0 = (\beta - \alpha)(\gamma - \alpha)(\gamma - \beta)$, thus in every case the determinant is given by $(\beta - \alpha)(\gamma - \alpha)(\gamma - \beta)$